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NOTE

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János Bolyai was one of the inventors of non-Euclidean geometry. According to his unpublished manuscripts, he also achieved remarkable results in other fields in mathematics. His work in number theory is of special interest because it contains ideas which have hitherto been attributed to other mathematicians. © 1999 Academic Press

Bolyai János a nem-euklidészi geometria egyik megalkotója. Feldolgozatlan kézirataínak tanúsága szerint a matematika más ágaiban is figyelemre méltó eredményeket ért el. Számelméleti munkái azért tarthatnak különös érdeklődésre számot, mivel olyan gondolatokat tartalmaznak, amelyeket eddig más matematikusoknak tulajdonítottak. © 1999 Academic Press

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1. INTRODUCTION

János Bolyai (1802–1860) was one of the founders of non-Euclidean geometry. During his life only his most important work, the *Appendix* [3], was published, although he left several thousand pages of manuscripts, which were stored in boxes in different places. Some of these have been lost, but fortunately most of them have been preserved. Thirteen thousand pages are now kept in the Teleki–Bolyai Library of Marosvásárhely [5].

Many authors have treated János Bolyai's life and activity. The most important of these are Paul Stäckel [24], Lajos Dávid [6], György Alexits [1], Barna Szénássy [25; 26], Tibor Weszely [27], Neumann, Salló, and Toró [20], and the authors of the volume of studies *János Bolyai's Life and Activity* [4]. Nevertheless, Bolyai's manuscripts have yet to be studied thoroughly.

These manuscripts—mostly fragments—do not constitute whole works and are written on various types of paper, envelopes, official documents, theater programs, etc. Bolyai did not write his notes with the public in mind; he wrote them for his own satisfaction. Thus, he did not perfect them, and his ideas do not always reach a conclusion. He created his own symbols and words; I have come across many difficult and unusual symbols. His texts contain lots of erasures, inserted words, and repetitions. For these reasons, it is difficult to make an inventory of his notes, mathematical and otherwise.

By deciphering and examining the many thousands of manuscripts kept in the Teleki–Bolyai Library, I have found many notes containing Bolyai's nongeometrical, and until now,

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unknown mathematical ideas. His number-theoretic research is of particular interest. Here, I will sketch Bolyai's research in the field of number theory.[1]

2. JÁNOS BOLYAI AND NUMBER THEORY

It is well known that János Bolyai worked on problems in number theory, but it is generally held that he accomplished nothing noteworthy in the field. My work on Bolyai's unpublished manuscripts suggests that this opinion needs to be corrected, however. To begin with, that Bolyai valued number theory and found the contemplation of number-theoretic results fascinating comes through clearly in various of his manuscripts. For example, he stated that "in number theory not only the problems of integral numbers can be found, but also the most important, most useful, most essential, most beautiful, most interesting, most charming problems of all mathematics" [5, 1179/24]. Moreover, although Bolyai admired the work of the premier early 19th-century number theorist, Carl Friedrich Gauss (1777–1855), he did not totally agree with the German's characterization of the field. "Gauss affirmed he had been engaged in number theory very early," Bolyai noted: "It remained his favorite branch of research until the end of his life, but he called it, mistakenly, the queen of mathematics" [5, 938/1^v].

In the summer of 1801, Gauss's fundamental number-theoretic work, *Disquisitiones arithmeticae*, was published. János Bolyai knew this work well and used it often. (Today, his copy of it—complete with his fading marginal notes—can be found in the Library of the Hungarian Scientific Academy.) Bolyai wrote that "[i]f there is anyone who wants to try his strength at the greatest and deepest work of the human mind. . . I suggest, for example, . . . Gauss's work, entitled *Disquisitiones arithmeticae*" [5, 845/7^v]. Elsewhere he offered the opinion that "the *Disquisitiones arithmeticae* . . . deserves eternal preservation, appreciation, and admiration" [5, 800/1^v].

3. THE CONVERSE OF FERMAT'S THEOREM

Like many others, János Bolyai tried to find a method for expressing any prime number in terms of a suitable formula; Paul Stäckel (1862–1919) mentioned these attempts in [24, 1:173]. At times, Bolyai felt he would succeed in solving these problems. In an undated letter to his father Farkas Bolyai (1775–1856), he wrote that "I have no doubt that I will manage to find the formula for prime numbers of any form very soon" [5, 800/1^v]. Here, Bolyai referred to a formula for prime numbers not only in the set of natural numbers but also in the ring of complex (Gaussian) integers, and he left notes concerning both problems among his papers.

Initially, Bolyai thought he could find the formula of prime numbers by means of the so-called "little theorem" of Pierre de Fermat (1601–1665), namely, if p is a prime number and a is an integer number which is not divisible by p, then

$$a^{p-1} \equiv 1 (\mathrm{mod}\ p). \tag{1}$$

The converse of Fermat's little theorem is not true, that is, if (1) is true, it does not necessarily follow that p is a prime number. In fact, for any natural number a, there are infinitely many

[1] For a fuller treatment, see [15; 16; 17; 18; 19].

composite numbers n such that

$$a^{n-1} \equiv 1 (\mathrm{mod}\, n). \tag{2}$$

For $a = 2$, a composite number n for which the congruence (2) is satisfied is called a pseudoprime number with respect to 2. Many mathematicians have studied these interesting numbers, but Bolyai was actually among the first to find the smallest pseudoprime number with respect to 2, a fact that has thus far gone unrecorded in the history of mathematics.

Bolyai's father had encouraged him to try to prove the converse of Fermat's little theorem, but after a few attempts János discovered that the converse is not true and found many composite numbers satisfying (2). In a letter written in May of 1855, he informed his father about the discovery that the number 341 is pseudoprime. He explained that

> Although I could not find my old investigations concerning $2^{(p-1)/2}$ that I promised [you] the day before yesterday, yesterday I realized ... that to resolve ... the doubt about the main problem ... it is enough to show ... that $2^{(m-1)/2} = 1 (\mathrm{mod}\, m)$ holds even if m is not prime, which can be proved by a single example, like the following, which I obtained after trying many numbers, but not by accident: $2^{340} - 1$ is divisible by $341 = 11 \cdot 31$, which we can obviously prove from the fact that $2^{10} = 1024$, which, after dividing by 341 gives 1 as remainder, so (Disq. Ar. 7 §) $(2^{10})^{17} = 2^{170} = 2^{(341-1)/2}$ and 2^{341-1} also give a remainder of 1 if divided by 341, so Fermat's theorem and the beautiful conjecture concerning the $2^{(m-1)/2}$ (which would be a nice criterion for prime numbers if it were valid) are generally not valid, even when $a = 2$. [5, 1018/1]

Bolyai thus proved that the composite number 341 really satisfies congruence (2) for $a = 2$. His proof can be found in other manuscripts as well [5, $1265/33^{v}$].

Consider, for example, a part of his calculation given in a manuscript he wrote in German (see Fig. 1):

> Ist $a^{p-1} \equiv 1$ O p, $a^{q-1} \equiv 1$ O q, also $a^{(p-1)(q-1)} \equiv 1$ O pq. Und es wird off., jedoch nur dann $a^{pq-1} \equiv 1$ O pq (sein), wenn $a^{pq-1} \equiv a^{(p-1)(q-1)} = a^{pq-p-q+1}$, oder $a^{(p-1)+(q-1)} = a^{p-1} \cdot a^{q-1} \equiv 1$ O pq ist. Nun ist $a^{p-1} = 1 + hp$, $a^{q-1} = 1 + kq$, also ... $a^{pq-1} \equiv 1$ O pq ist #: $(1+hp)(1+kq) \equiv 1$, das ist $hp + kq \equiv C$ O pq, $[(a^{p-1} - 1) + (a^{q-1} - 1)]/p$. Da nun hier $(a^{p-1} - 1)/p = h$, $(a^{q-1} - 1)/q = k$ ist: so setzen wir od, nehmen an, q sei ein + Prim-Masz > 1 von h: so ist einmal $(a^{p-1} - 1)/pq$ ganz; ... wobei auch $(a^{q-1} - 1)/pq$ ganz wird. Fangen wir mit den kleisten Unpar Werthen von p an, ... , und finden wir $2^{340} \equiv 1$ O 341. [5, $1265/33^{v}$]

In modern notation and terminology, this translates as:

> Because $a^{p-1} \equiv 1 (\mathrm{mod}\, p)$ and $a^{q-1} \equiv 1 (\mathrm{mod}\, q)$, consequently $a^{(p-1)(q-1)} \equiv 1 (\mathrm{mod}\, pq)$. It is evident, that the congruence $a^{pq-1} \equiv 1 (\mathrm{mod}\, pq)$ is not fulfilled, unless $a^{pq-1} \equiv a^{(p-1)(q-1)} = a^{pq-p-q+1}$, or $a^{(p-1)+(q-1)} = a^{p-1} \cdot a^{q-1} = 1 (\mathrm{mod}\, pq)$. Now we have $a^{p-1} = 1 + hp$, $a^{q-1} = 1 + kq$, so ... $a^{pq-1} \equiv 1 (\mathrm{mod}\, pq) \Leftrightarrow (1+hp)(1+kq) \equiv 1$, that is, $hp + kq \equiv 0 (\mathrm{mod}\, pq)$, $[(a^{p-1} - 1) + (a^{q-1} - 1)]/pq$. Because here we have $(a^{p-1} - 1)/p = h$, $(a^{q-1} - 1)/q = k$, then by assuming that q is a prime divisor of h greater then 1: we have that $(a^{p-1} - 1)/pq$ is an integer; ... where $(a^{q-1} - 1)/pq$ is also an integer. Starting with the smallest odd values of p, ... , and we reach $2^{340} \equiv 1 (\mathrm{mod}\, 341)$.

Bolyai thus found the smallest pseudoprime number with respect to 2 this way. His conditions for $a = 2$ can be written as

$$2^{p-1} \equiv 1 (\mathrm{mod}\, q) \qquad \text{and} \qquad 2^{q-1} \equiv 1 (\mathrm{mod}\, p).$$

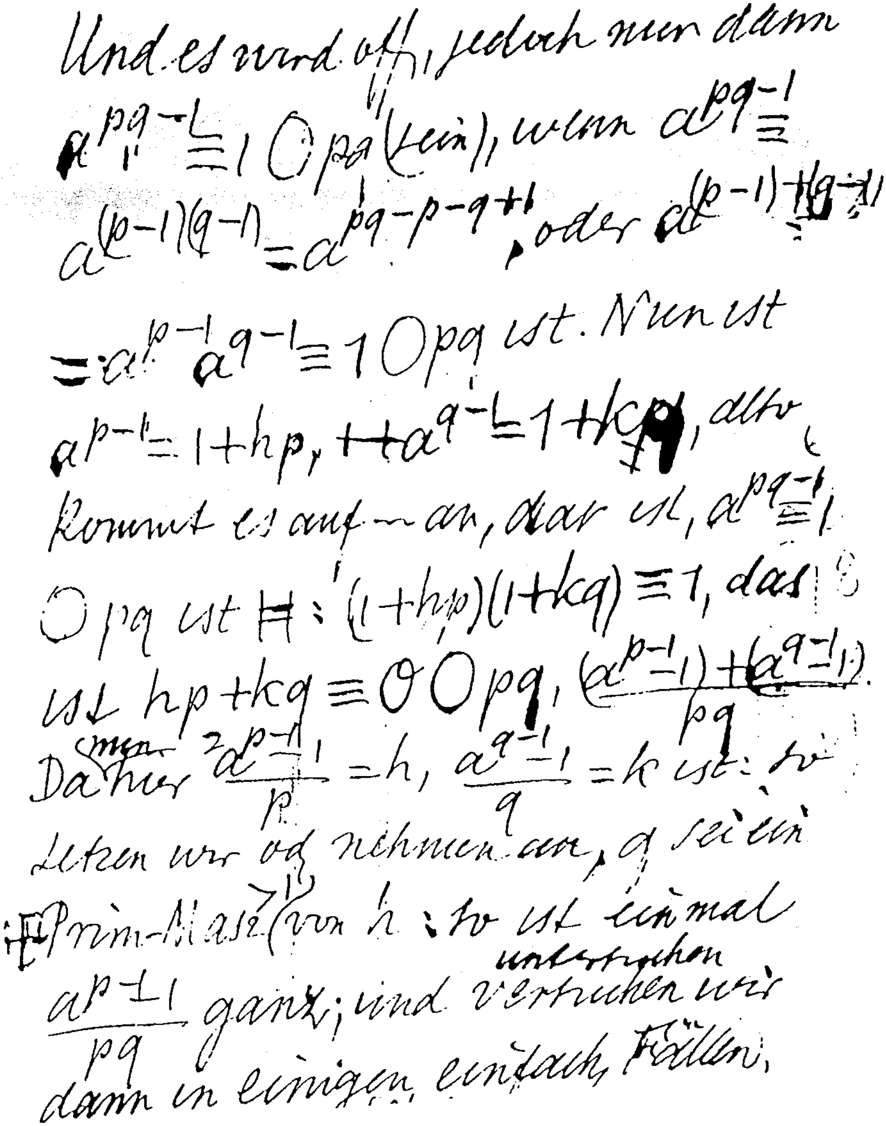

Und es wird oft, jedoch nur dann
$a^{pq-1} \equiv 1 \,\mathrm{O}\, pq$ (sein), wenn $a^{pq} \equiv$
$a^{(p-1)(q-1)} = a^{pq-p-q+1}$, oder $a^{(p-1)+(q-1)}$
$= a^{p-1} a^{q-1} \equiv 1 \,\mathrm{O}\, pq$ ist. Nun ist
$a^{p-1} = 1+hp$, $a^{q-1} = 1+kq$, also
kommt es auf — an, dass ist, $a^{pq} \equiv 1$
$\mathrm{O}\, pq$ ist: $(1+hp)(1+kq) \equiv 1$, das
ist $hp+kq \equiv 0 \,\mathrm{O}\, pq$, $\frac{(a^{p-1}-1)+(a^{q-1}-1)}{pq}$.
Da hier (wenn) $\frac{a^{p-1}-1}{p} = h$, $\frac{a^{q-1}-1}{q} = k$ ist: so
setzen wir od nehmen an, q sei ein
Prim-Maß von h: so ist einmal
$\frac{a^{p-1}-1}{pq}$ ganz; und versuchen wir (untersuchen)
dann in einigen einfach Fällen;

FIGURE 1

He knew that these conditions imply that

$$2^{pq-1} \equiv 1 (\operatorname{mod} q).$$

(This latter fact, now known as Jeans's theorem [14], was published in 1898 by the physicist James Jeans (1877–1946), several decades after Bolyai's death.) Bolyai then tried small values of p and found possible values of q from $2^{p-1} \equiv 1 (\operatorname{mod} q)$. He checked to see if these values of q satisfied $2^{q-1} \equiv 1 (\mathrm{mod}\, p)$. There is a surprising similarity here between Bolyai's thoughts and an idea applied by Paul Erdős (1913–1996) in a note published in 1949 [9].

János Bolyai had been engaged with this problem for a very long time. Although, as he emphasized in the letter to his father quoted above, one example is enough to refute the converse of Fermat's little theorem, he left other counterexamples in his papers. For example, he established the congruences

$$4^{14} \equiv 1 (\mathrm{mod}\, 15), \qquad 5^3 \equiv 1 (\mathrm{mod}\, 4),$$

and

$$2^{2^{32}} \equiv 1 (\mathrm{mod}\, 2^{32} + 1 = 641 \cdot 6700417). \tag{3}$$

Bolyai thus focused on counterexamples of the converse of Fermat's little theorem at the time when hardly any other mathematicians were working on this subject. In his *History of the Theory of Numbers*, Leonard Eugene Dickson (1874–1954) mentioned that the number 341 had already been found in 1830 by an anonymous author [2], and that Frédéric Sarrus [21] had found the congruence $2^{170} \equiv 1 (\mathrm{mod}\, 341)$ in 1820 [7, 1:92]. An examination of [2, 21] reveals that although there were a few common ideas, Bolyai's method was different. It seems clear that Bolyai did not know of this earlier work. Note, too, that the earliest occurrences recorded thus far of congruences such as (3), in which the numbers of Fermat

$$F_k = 2^{2^k} + 1 \tag{4}$$

appear, were published in early 20th-century works [7, 1:94]. Bolyai was thus the first to prove that F_5 is a pseudoprime number.

Moreover, in a manuscript page [5, 1193/20^v], Bolyai set himself the task of finding the conditions for the validity of the following congruence:

$$a^{pqr-1} \equiv (\mathrm{mod}\, pqr),$$

where p, q, r are prime numbers and a is an integer not divisible by p, q, or r. This time he could not solve the problem, but his experiment suggests that he may have been at work on a generalization of Jeans's theorem [15].

4. GAUSSIAN INTEGERS

János Bolyai did not succeed in finding a formula for prime numbers in the ring of integer numbers. However, his attempts to describe the complex prime numbers in the ring of complex integers were successful. It is interesting that he wrote a paper on complex numbers entitled *Responsio*, published posthumously by Stäckel in 1899 [23], but this paper contains nothing about complex integers.

The notion of a complex integer number was apparently first introduced by Gauss in his papers [10; 11]. Independent of Gauss and at about the same time, however, Bolyai also worked out the divisibility properties of the complex integers. In a letter to his father dated in 1845, Bolyai wrote that “I have searched for the theory of imaginarians at its own place, and I found it in 1831” [5, 740/1]. In another undated letter to his father, he mentioned “the basic properties of prime numbers, extended for imaginarians a long time ago by me” [5, 800/3]. And finally in a letter on complex integers written in his old age—around 1855—also to his father, he stated that “I completely clarified this subject about a quarter century ago” [5, 982/8^v].

Further evidence of the younger Bolyai’s independent discovery can be found in the letters that Farkas Bolyai and Gauss wrote to each other [22]. In a letter addressed March 6, 1832 to his “unforgettable friend,” Gauss called the elder Bolyai’s attention to his work [10] and pointed out that “you find in it, expounded in a few pages, my view about the imaginary quantities” [22, 102]. It is strange that Gauss failed to mention his work [11], which contains the complete theory of complex integers, despite the fact that Farkas Bolyai had asked him many times for his detailed work on imaginary quantities. Since Gauss had still not complied with his request by 1848, Bolyai wrote resignedly that “I have been waiting for the exposition of your theory of imaginary numbers for a long time, and I give up all hope” [22, 129].

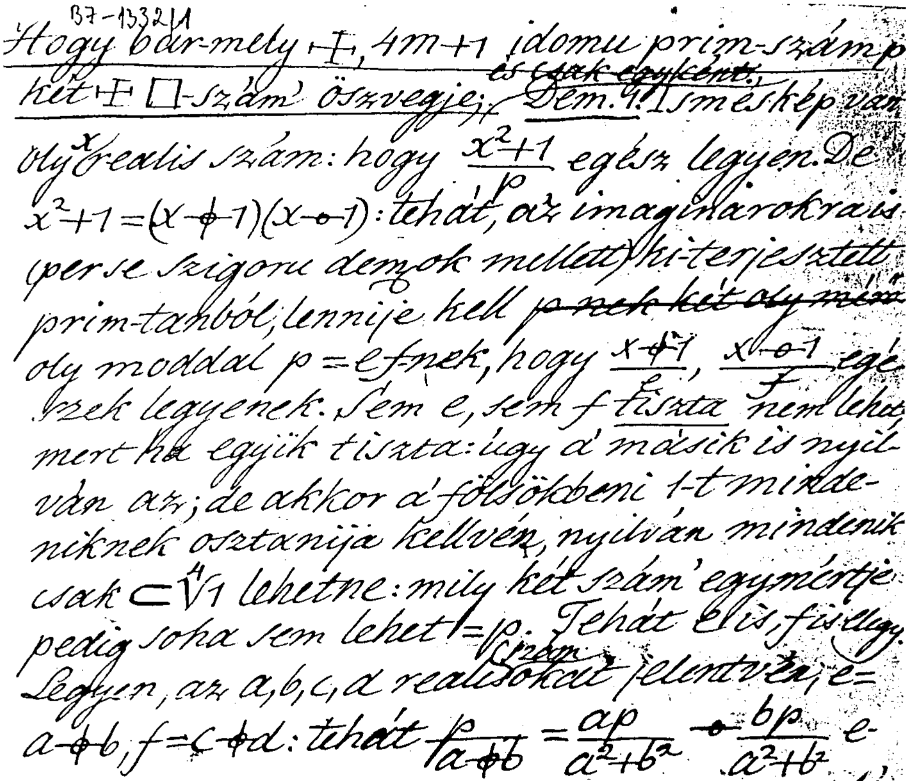
After the 1831 volume of the *Göttingische gelehrte Anzeigen* (which contains Gauss’s more abbreviated paper [10]) finally arrived in Marosvásárhely[2] János Bolyai read the article [10] and quoted it in many places, although he was still unaware of Gauss’s more important work [11]. This fact is confirmed by a comparison of the writings of the two mathematicians. Bolyai’s paper did not contain the definitions of several key concepts (for example, that of the associated element or norm of complex numbers) used by Gauss, and its proofs differ from Gauss’s. Moreover, Bolyai’s theory was not as exhaustive as what Gauss presented in [11]; it was also not written as a completed whole, but rather as results scattered through his manuscript notes. Without going into detail (for which see [19]), suffice it to say that Bolyai answered all the fundamental questions concerning the divisibility of complex integers (the definition of complex integers, prime factorization, congruences) and, in later life, successfully applied his results in the proofs of several theorems in number theory. For example, he worked out at least four simple proofs of Fermat’s two-square theorem using complex integers. (For three of them, see [17].)

In his very short fourth proof, Bolyai started from the known theorem that if p is a prime number of the form $4k+1$, then there exists an integer x such that $(x^2+1)/p$ is also an integer. He wrote this fraction in the form $(x+i)(x-i)/p$ and first proved that $p=(a+bi)(c+di)$, where a, b, c, d are nonzero integers. The theorem then followed as an easy consequence [5, 1332/1].

Part of his proof, written in the original Hungarian (see Fig. 2), follows:

> Hogy bármely $+$, $4m+1$ idomu prím-szám p két $+$ □-szám öszvegje; és csak egyként. Dem. 1. Isméskép van oly x realis szám: hogy $(x^2+1)/p$ egész legyen. De $x^2+1=(x\oplus 1)(x\ominus 1)$: tehát, az imaginárokra is (per se szigorú dem. ok mellett) ki-terjesztett prim-tanból, lennie kell egy olyan mod-dal $p=ef$-nek, hogy $(x\oplus 1)/e$, $(x\ominus 1)/f$ egészek legyenek. Sem e, sem f tiszta nem lehet, mert ha egyik tiszta: úgy a másik is nyilván az: de akkor á fölsökbeni 1-t mindeniknek osztanija kellvén, nyilván mindenik csak $\subset \sqrt[4]{1}$ lehetne: mily két szám egymértje pedig soha sem lehet $=p$. Tehát e is, f is elegy. Legyen az a, b, c, d realis számokat jelentvén; $e=a\oplus b$, $f=c\oplus d$: tehát $p/(a\oplus b) = ap/(a^2+b^2)\ominus(bp/(a^2+b^2))$ e $-$ (gesz). [5, 1332/1]

[2] It still is in the Teleki–Bolyai Library.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Hogy bár-mely +, 4m+1 idomu prim-szám p két + □-szám' összvegje; ~~és csak egyféleképp~~. Dem. 4. Ismeretes hogy van oly x reális szám: hogy $\frac{x^2+1}{p}$ egész legyen. De $x^2+1=(x+i)(x-i)$: tehát, az imaginarokra is (per se szigoru demok mellett) ki-terjesztett prim-tanból, lennie kell ~~p-nek két oly mini~~ oly moddal $p=ef$-nek, hogy $\frac{x+i}{e}$, $\frac{x-i}{f}$ egészek legyenek. Sem e, sem f tiszta nem lehet: mert ha egyik tiszta: ugy a' másik is nyilván az; de akkor a' fölsőkbeni 1-t mindeniknek osztania kellvén, nyilván mindenik csak $\subset \sqrt[4]{1}$ lehetne: mily két szám' egymértje pedig soha sem lehet $= p$. Tehát e is, f is elegy. Legyen, az a, b, c, d reálisokat jelentvén, $e=a+ib$, $f=c+id$: tehát $\frac{p}{a+ib}=\frac{ap}{a^2+b^2}-i\frac{bp}{a^2+b^2}$ e-

FIGURE 2

Using modern notation, this translates as:

> Any positive prime number of the form $4m+1$ can be written uniquely as the sum of two squares. *Proof*. It is known, that such a number x, for which $(x^2+1)/p$ is an integer, exists. But $x^2+1=(x+i)(x-i)$, so by using the divisibility properties generalized for complex numbers (beside rigorous proofs) it can be written in the form $p=e\cdot f$ in such a way, that $(x+i)/e$ and $(x-i)/f$ are integers. Here neither e nor f can be pure imaginary numbers, because if one of them were so, then the other would also necessarily be so. But then they would divide 1, which is possible only if they are equal to $\pm 1, \pm i$. But in this case the product of two numbers cannot be equal to p. Consequently both e and f are mixed complex numbers, that is, $e=a+ib$, $f=c+id$, where a, b, c, d are integers. So $p/(a+ib)=ap/(a^2+b^2)-i(bp/(a^2+b^2))$ in – (teger).

But then $p=(a-bi)(c-di)$, too, which implies $p^2=(a^2+b^2)(c^2+d^2)$. Because $a^2+b^2>1$, $c^2+d^2>1$, we have $p=a^2+b^2=c^2+d^2$, as desired [5, 1333/1^v]. In 1844, Gotthold Eisenstein (1823–1852) also proved Fermat's two-square theorem using complex integers [7, 2:236; 8], but in a different way.

5. FERMAT NUMBERS

Later in his life, János Bolyai worked hard to determine whether the Fermat number F_6 is prime. His attempts did not lead to the desired result, but he did prove that all Fermat numbers F_k are of the form $6n - 1$ and so are not divisible by 3. He also proved that $2^{64} \equiv 1(\bmod 3)$. As is well known, in 1732 Leonhard Euler (1707–1783) proved that F_5 is a composite number. In 1880 Fortune Landry proved the same thing for the next Fermat number, F_6 [7, 1:377]. The manuscripts thus show that Bolyai also worked on this problem more then 20 years earlier than Landry.

6. FINAL NOTES

Thanks to Stäckel's pioneering work in Bolyai's archives, many of János Bolyai's unpublished mathematical results became known around 1900. Bolyai had lived at the fringe of the international mathematical world and had had personal connections with only one mathematician, his father Farkas Bolyai. János had thus developed not only his geometrical system independently of others but essentially all of his mathematical ideas. As I hope the present study has shown, the ongoing investigation of Bolyai's manuscripts has much to yield; Bolyai made contributions to number theory worthy of note.

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REFERENCES

1. György Alexits, *János Bolyai's World*, Budapest: Akadémiai Kiadó, 1977. [In Hungarian]
2. Anonymous, Théorèmes et problèmes sur les nombres, *Journal für die reine und angewandte Mathematik* **6** (1830), 100–106.
3. János Bolyai, *Appendix, Scientiam spatii absolute . . .*, Marosvásárhely, 1832.
4. *János Bolyai's Life and Activity*, Bucharest: Állami Tudományos Könyvkiadó, 1953. [In Hungarian]
5. János Bolyai, *manuscripts*, Târgu-Mureş: Teleki-Bolyai Library.
6. Lajos Dávid, *The Life and Activity of the Two Bolyais*, Budapest: Gondolat, 1979. [In Hungarian]
7. Leonard E. Dickson, *History of the Theory of Numbers*, reprint, New York: Chelsea, 1971.
8. Gotthold Eisenstein, Beiträge zur Kreistheilung, *Journal für die reine und angewandte Mathematik* **27** (1844), 269–278.
9. Paul Erdős, On the Converse of Fermat's Theorem, *American Mathematical Monthly* **56** (1949), 623–624.
10. Carl Friedrich Gauss, Theoria residuorum biquadraticorum: Commentatio secunda, *Göttingische gelehrte Anzeigen*, Stück 64, 1831, 625–638, reprinted in [12, 169–178].
11. Carl Friedrich Gauss, Theoria residuorum biquadraticorum: Commentatio secunda, *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores* **7** (1832), 89–148, reprinted in [12, 93–148], German translation in [13, 534–586]
12. Carl Friedrich Gauss, *Werke*, vol. 2, Göttingen: Teubner, 1863.
13. Carl Friedrich Gauss, *Untersuchungen über höhere Arithmetik*, trans. Hans Maser, Berlin: Springer-Verlag, 1889; reprint ed., New York: Chelsea, 1965.
14. James Hopwood Jeans, The Converse of Fermat's Theorem, *Messenger of Mathematics* **27** (1897–1898), 174.
15. Elemér Kiss, Fermat's Theorem in János Bolyai's Manuscripts, *Mathematica Pannonica* **6** (1995), 237–242.
16. Elemér Kiss, János Bolyai, the First Hungarian Number Theorist, *Mathematical and Physical Journal for Secondary School (KöMaL)*, Special English Language Issue, July 1996, 8–10.
17. Elemér Kiss, János Bolyai's Examinations of the Decomposition of Prime Numbers of the Form $4m + 1$ into the Sum of Two Squares, *Polygon* **6**(2) (1996), 1–11. [In Hungarian]
18. Elemér Kiss, A Short Proof of Fermat's Two-Square Theorem Given by János Bolyai, *Mathematica Pannonica* **8**(2) (1997), 293–295.

19. Elemér Kiss, János Bolyai's Investigations in the Theory of Complex Integers, *Polygon* **8**(1) (1998), 1–11. [In Hungarian]
20. Mária Neumann, Ervin Salló, and Tibor Toró, *From Nothing I Have Created a New World*, Temesvár: Facla Könyvkiadó, 1974. [In Hungarian]
21. Frédéric Sarrus, Démonstration de la fausseté du théorème énoncé á la page 320 du IX^{e} volume de ce recueil, *Annales de mathématiques* **10** (1812–1820), 184–187.
22. Franz Schmidt and Paul Stäckel, *Briefwechsel zwischen Carl Friedrich Gauss und Wolfgang Bolyai*, Leipzig: Teubner, 1899.
23. Paul Stäckel, The Theory of Imaginary Numbers in the Bequested Manuscripts of János Bolyai, *Mathematikai és Természettudományi Értesítő*, **17** (1899), 259–292. [In Hungarian]
24. Paul Stäckel, *Geometrical Examinations of Farkas Bolyai and János Bolyai*, 2 vols., Budapest: Kiadja a Magyar Tudományos Akadémia, 1914. [In Hungarian]
25. Barna Szénássy, *Bolyai János*, Budapest: Akadémiai Kiadó, 1978. [In Hungarian]
26. Barna Szénássy, *History of Mathematics in Hungary until the 20th Century*, Budapest: Akadémiai Kiadó, 1992.
27. Tibor Weszely, *Mathematical Activity of János Bolyai*, Bucharest: Kriterion Könyvkiadó, 1981. [In Hungarian]